First-Order Logic

Relations

Motivation, Again

We are now in position to translate the Monty Python argument.

The argument sketch is a Monty Python sketch.

Every Monty Python sketch is funny.

Therefore, the argument sketch is funny.

Motivation, Again

Here's how …

a = the argument sketch

F = … is funny

M = … is a Monty Python sketch

Motivation, Again

Given our notational choices, we now have …

The argument sketch is a Monty Python sketch.

Every Monty Python sketch is funny.

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Motivation, Again

Given our notational choices, we now have …

Ma

Every Monty Python sketch is funny.

Therefore, the argument sketch is funny.

Motivation, Again

Given our notational choices, we now have …

Ma

$(\forall x)(Mx \rightarrow Fx)$

Therefore, the argument sketch is funny.

Motivation, Again

Given our notational choices, we now have …

$$\frac{\begin{array}{l}Ma \\ (\forall x)(Mx \rightarrow Fx)\end{array}}{Fa}$$

Motivation, Again

Given our notational choices, we now have …

Ma

$(\forall x)(Mx \rightarrow Fx)$

Fa

Now we need to either develop a semantics for such sentences (and test for validity) or we need to develop a proof theory.

More Categoricals

Let’s translate a *particular* categorical sentence: “Some Muppets wear hats.”

More Categoricals

What does it mean to say that *some* Muppets wear hats?

I can find something that is both a Muppet and a hat-wearer.

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> I can find something that is both a Muppet and a hat-wearer.

NO

NO

NO

More Categoricals

What does it mean to say that *some* Muppets wear hats?

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NO

NO

NO

More Categoricals

What does it mean to say that *some* Muppets wear hats?

I can find something that is both a Muppet and a hat-wearer.

NO

NO

NO

YES!

More Categoricals

How do we translate the sentence, “Some Muppets wear hats”?

Let’s start with the predicates:

M = “… is a Muppet”

H = “… wears a hat”

More Categoricals

Some Muppets wear hats.

$$(\exists x)(Mx \land Hx)$$

More Categoricals

Some Muppets wear hats.

$(\exists x)(Mx \wedge Hx)$

There is at least one x

More Categoricals

Some Muppets wear hats.

$$(\exists x)(Mx \wedge Hx)$$

There is at least one x

x is a Muppet

More Categoricals

Some Muppets wear hats.

$$(\exists x)(Mx \wedge Hx)$$

There is at least one x

x is a Muppet

and

More Categoricals

Some Muppets wear hats.

$$(\exists x)(Mx \land Hx)$$

There is at least one x

x is a Muppet

and

x wears a hat

More Categoricals

Some Muppets wear hats.

$$(\exists x)(Mx \wedge Hx)$$

There is at least one x

x is a Muppet

and

x wears a hat

More Categoricals

Some Muppets wear hats.

$(Ma \wedge Ha) \vee$

$(Mb \wedge Hb) \vee$

$(Mc \wedge Hc) \vee$

$\vdots$

One way to think of an existential quantifier is as a big disjunction.

Relations

First-order logic gives us the power to represent categorical sentences. But it is considerably more powerful than that! First-order logic also lets us represent *relational* claims.

Relations

A categorical sentence uses only one-place predicates and a single quantifier expression.

But often, we want to talk about *relations* between things.

Relations

One-place predicates are called *monadic predicates*. By contrast, *relations* have two or more places:

L = "__ loves __"

E = "__ eats more __ than __"

A = "__ asks __ to do __ for __"

S = "__ is shaking hands with __"

Relations

Attaching constant terms to a relation creates a simple sentence:

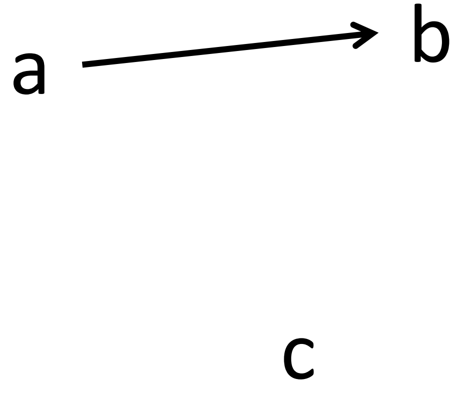
Lpk = “p loves k”

Dab = “a is one meter from b”

Smn = “m is shaking hands with n”

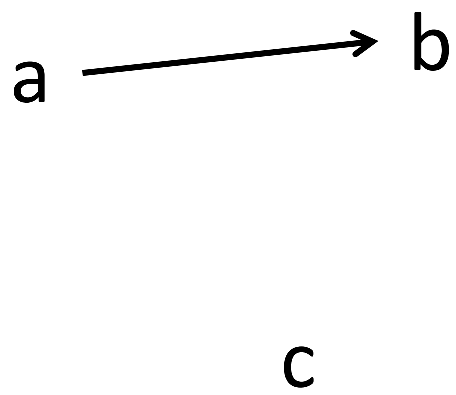
Relations

We will represent two-place relations using directed graphs. If Rab is true for constants a and b, then we draw an arrow from a to b.



Relations

Since there is no arrow from b to c, Rbc is false according to our picture.



Quantifiers and Relations

Adding quantifiers, we can translate more complicated sentences, like:

Betty is shaking hands with someone.

$(\exists x)Sbx$

Everyone loves Betty.

$(\forall x)Lxb$

Quantifiers and Relations

Given that L = "__ loves __" how should we translate the following sentence into English?

$$(\forall x)(\exists y)Lxy$$

Quantifiers and Relations

Given that L = "__ loves __" how should we translate the following sentence into English?

$$(\forall x)(\exists y)Lxy$$

Everyone loves someone or other.

Quantifiers and Relations

Given that L = "__ loves __" how should we translate the following sentence into English?

$$(\exists x)(\forall y)Lxy$$

Does the sentence above say the same thing as our earlier sentence?

$$(\forall y)(\exists x)Lxy$$

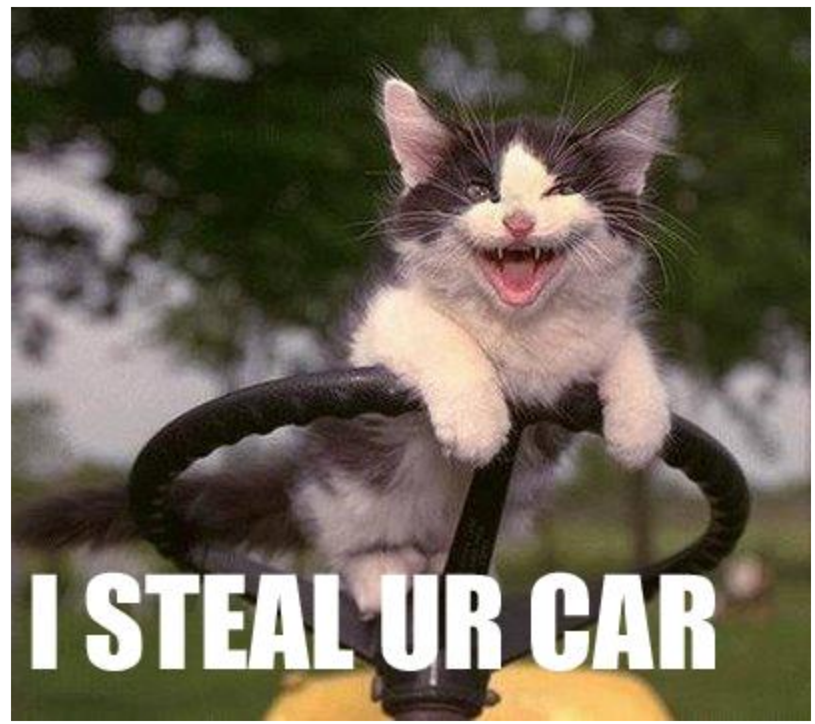
Quantifiers and Relations

A joke: Every 30 seconds, someone in the U.S. steals a car.

Quantifiers and Relations

A joke: Every 30 seconds, someone in the U.S. steals a car.

We have to find this person (or cat) and stop him!



Quantifier and Relations

Given that Lxy = x loves y, translate the following:

$(\forall x)(\exists y)Lxy$	$(\forall y)(\exists x)Lxy$
$(\exists x)(\forall y)Lxy$	$(\exists y)(\forall x)Lxy$

Interesting Relations

Some relations have special properties that we care about. We focus on three such properties:

Reflexive

Symmetric

Transitive

Interesting Relations

A relation R is *reflexive* just in case everything is R-related to itself.

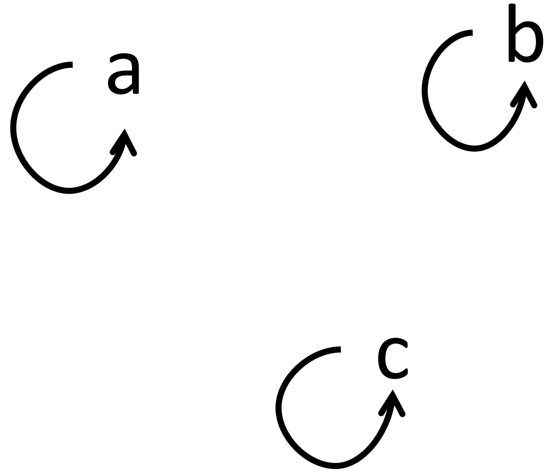
F = "__ is less than five meters from __"

Q = "__ is exactly as frustrating as __"

Interesting Relations

A relation R is *reflexive* just in case everything is R-related to itself.

$(\forall x)Rxx$



Interesting Relations

Some relations are *symmetric*. They have the same truth value regardless of the order of their inputs.

S = "__ is shaking hands with __"

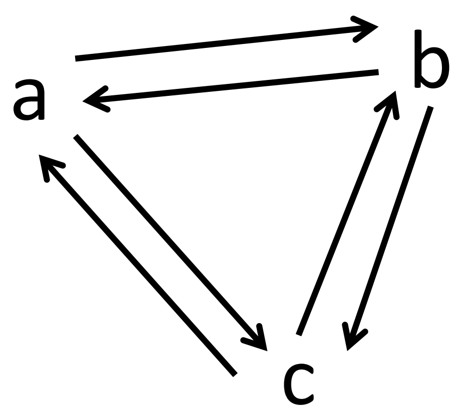
H = "__ is exactly as heavy as __"

N = "__ is nearby __"

Interesting Relations

A relation R is *symmetric* iff every pair that is R-related in one order is R-related in both orders.

$$(\forall x)(\forall y)(Rxy \rightarrow Ryx)$$



Interesting Relations

A relation R is *transitive* just in case when both Rab and Rbc hold, Rac holds as well.

W = "__ is heavier than __"

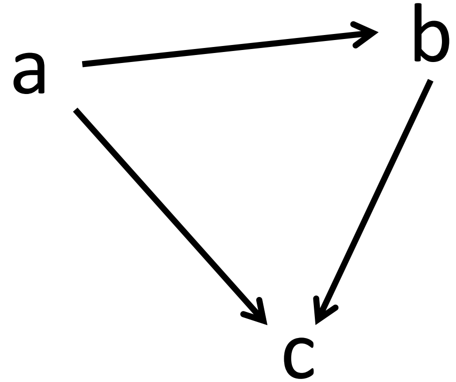
P = "__ is provable from __"

G = "__ is as green as __"

Interesting Relations

A relation R is *transitive* iff having Rab and Rbc guarantees having Rac.

$$(\forall x)(\forall y)(\forall z)((Rxy \wedge Ryz) \rightarrow Rxz)$$



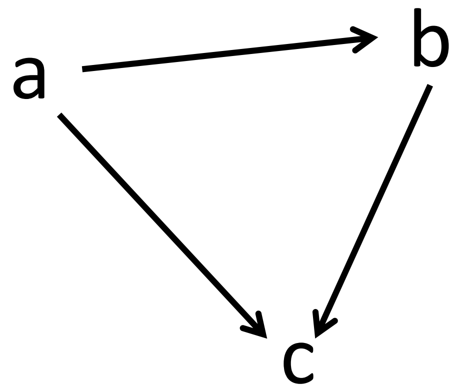
Interesting Relations

Some relations are reflexive, symmetric, *and* transitive. Such relations are called *equivalence relations*. The identity relation is an example of an equivalence relation.

Identity is such a special relation that we will give it its own symbol, =, and we will write (a = b), rather than =ab.

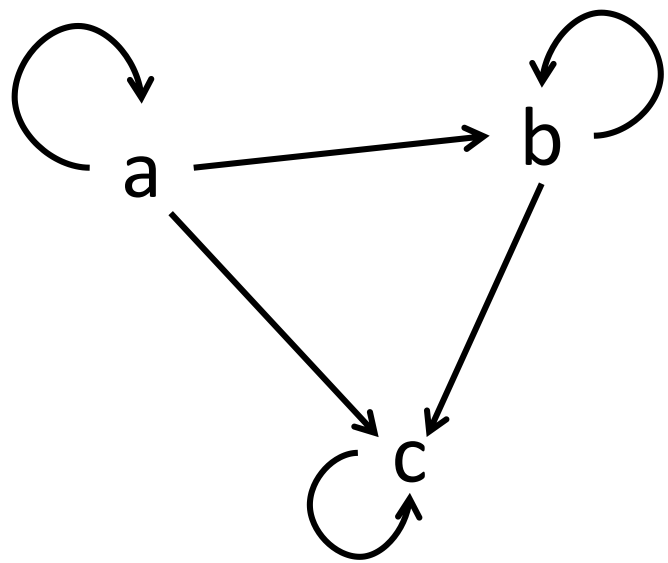
Interesting Relations

Suppose we have a relation R over three individuals as pictured below. What properties hold for the relation R?



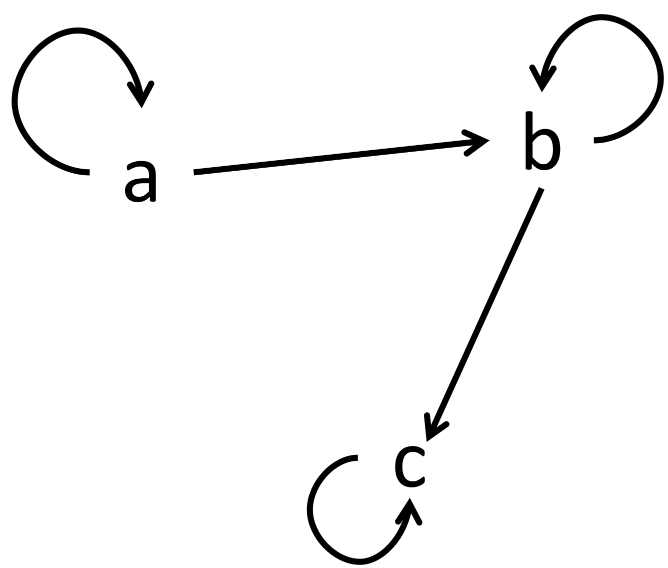
Interesting Relations

What properties hold for the relation now?



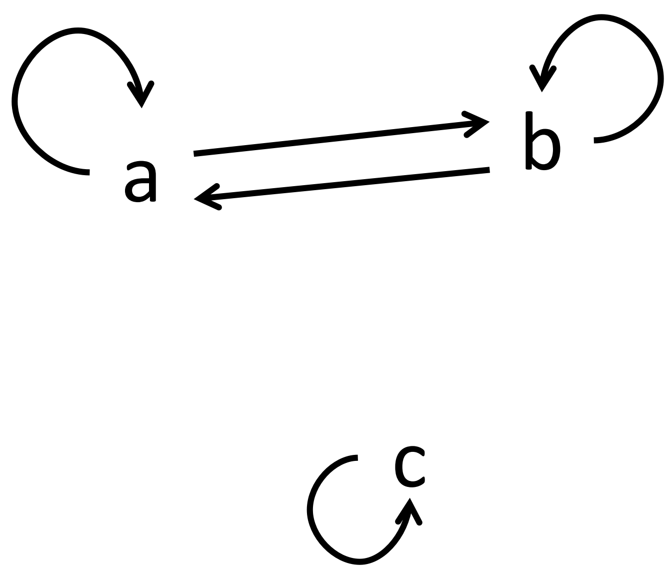
Interesting Relations

After removing the a to c edge, what properties hold for the relation?



Interesting Relations

Finally, what properties hold for the relation, now?



Interesting Relations

Let's do some simple translations. Suppose T = "… is taller than …" and b stands for Betty.

$(\exists x)Txb$

$(\forall x)Tbx$

Everyone is taller than someone or other.

Betty is not taller than herself.

Identity

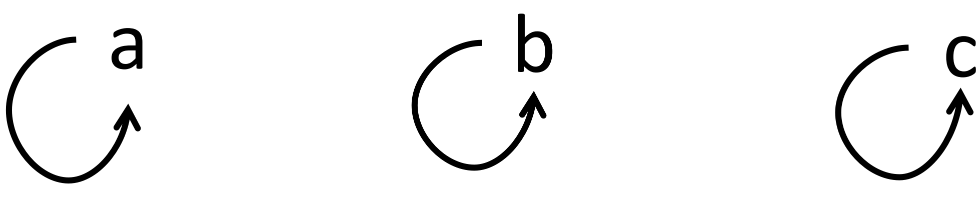
For the most part, we treat relations in a generic way. However, one relation is special.

Identity gets its own symbol, =, and we write (a = b), rather than =ab.

Identity

Identity is an equivalence relation: it is reflexive, symmetric, and transitive.

In fact, identity is the smallest or most fine-grained equivalence relation.



Identity

We can use identity to translate sentences involving superlatives or numerical claims.

Jim is the shortest man in the room.

$$(Mj \land Rj) \land (\forall x)((Mx \land Rx) \to (Sjx \lor (j = x)))$$

There is exactly one fish.

$$(\exists x)(Fx \land (\forall y)(Fy \to (y = x)))$$

Identity

Let’s try two more examples:

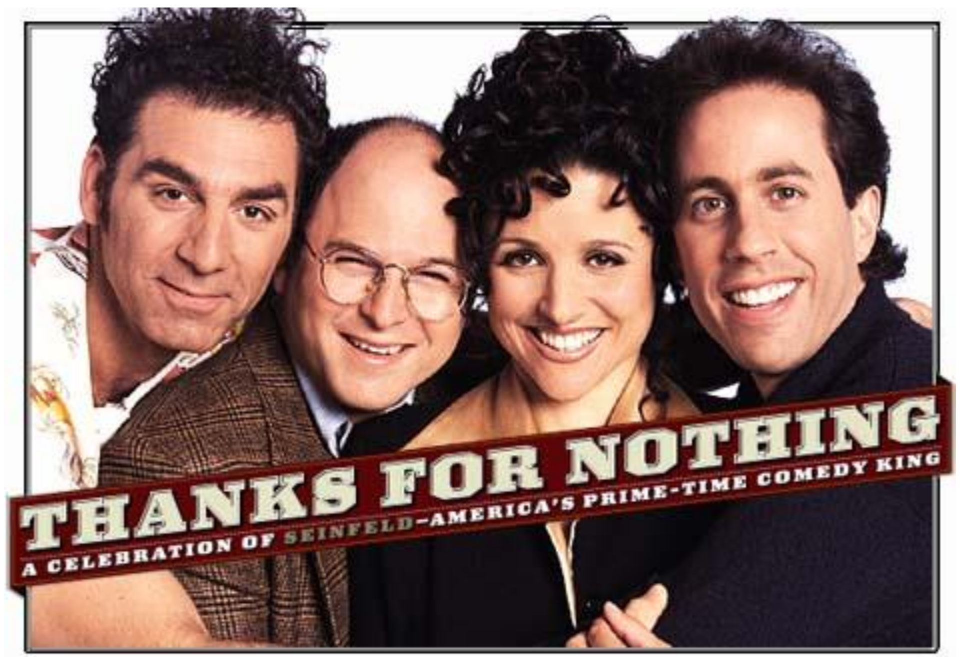
The Godfather was the best film of 1972.

There is exactly one instructor for PHIL 103.

A Brief Word About Nothing

Suppose you want to translate sentences like:

Seinfeld is a show about nothing.



A Brief Word About Nothing

When I want to translate sentences involving words like *nothing*, *nobody*, or *nowhere*, I will generally use the construction $\sim(\exists x)\phi$.

There isn't even one thing that would make ϕ true.

A Brief Word About Nothing

In the *Seinfeld* case, we will let A = "… is about ---" and S = "… is a show." Then let n denote the show *Seinfeld*. Then we can translate the sentence, "*Seinfeld* is a show about nothing," as follows:

$$(Sn \land \sim(\exists x)Anx)$$

A Brief Word About Nothing

Lewis Carroll (aka Charles Dodgson) made comic use of nothing in *Through the Looking Glass*.

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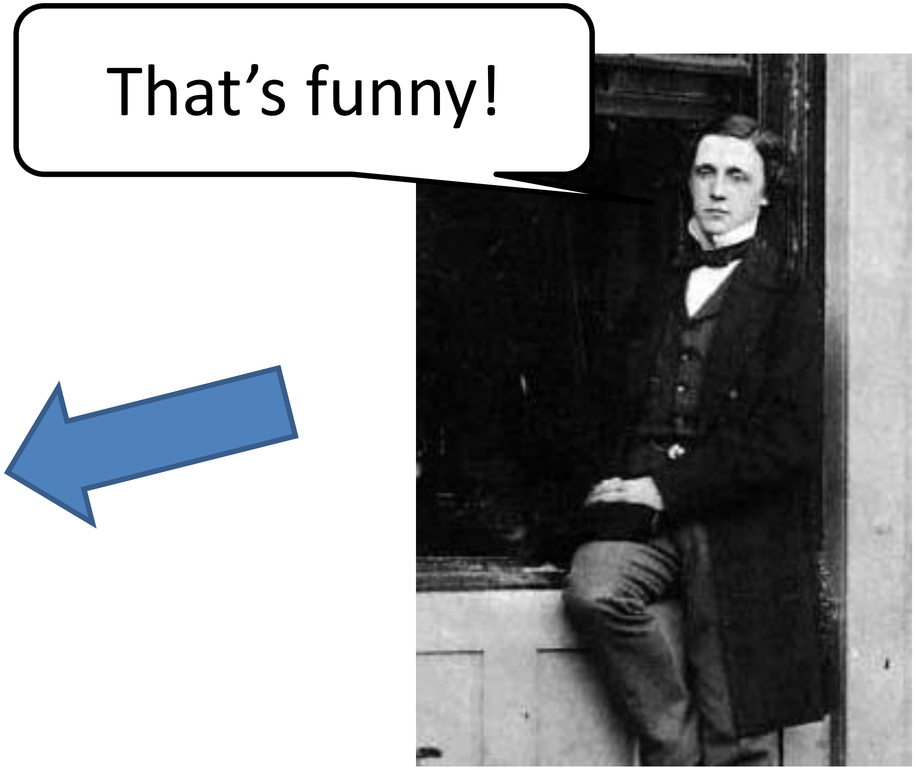
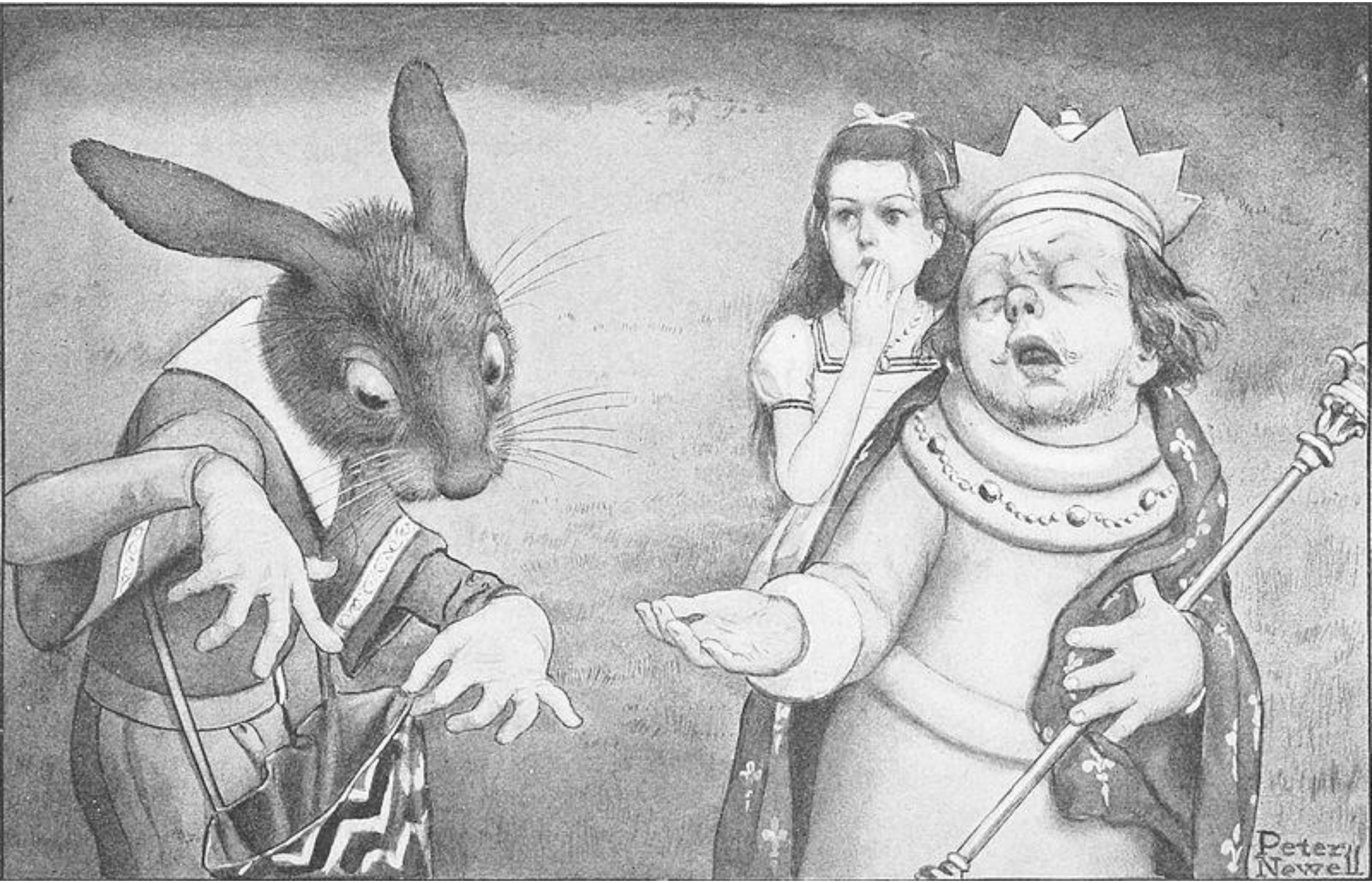
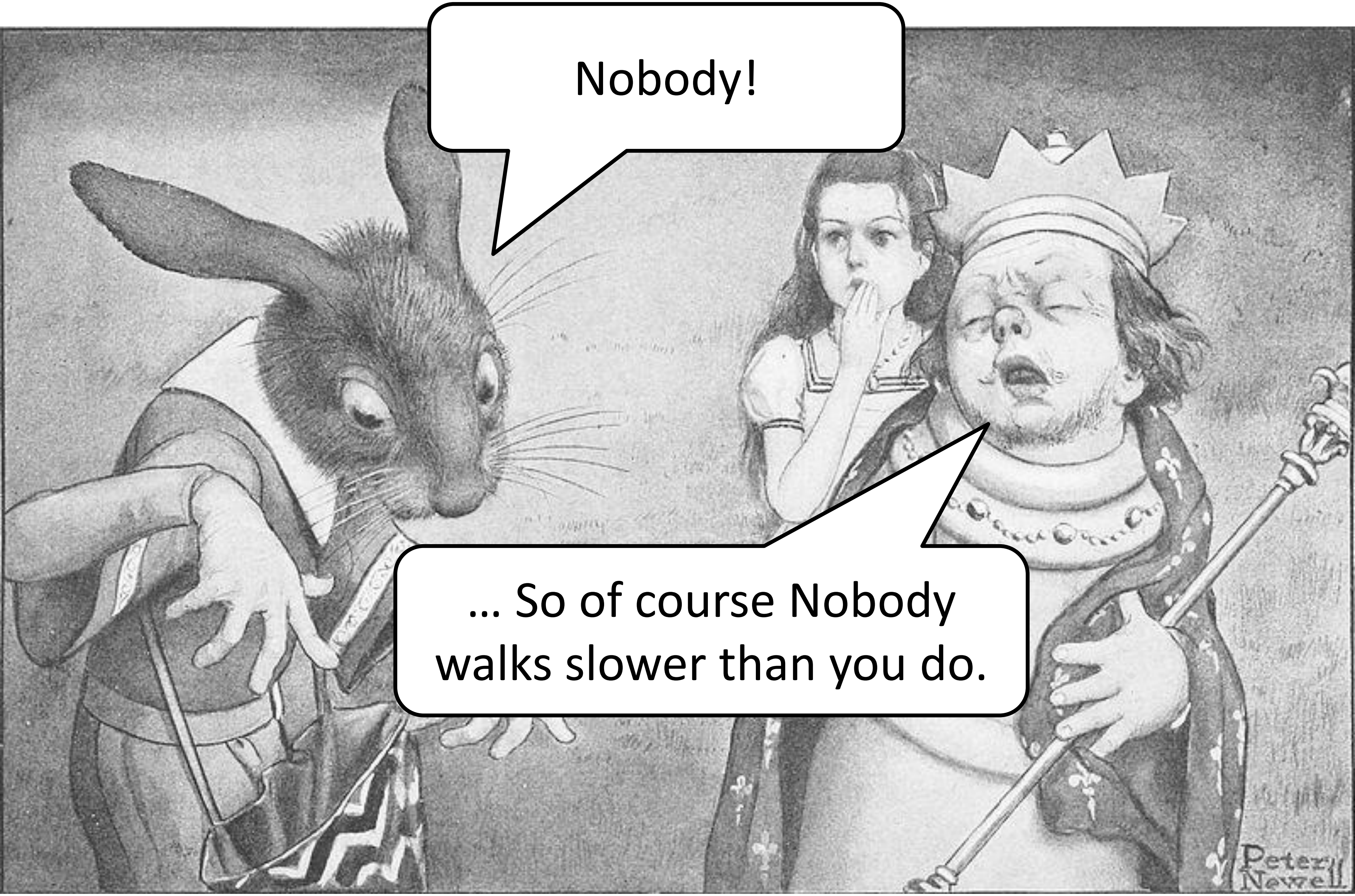
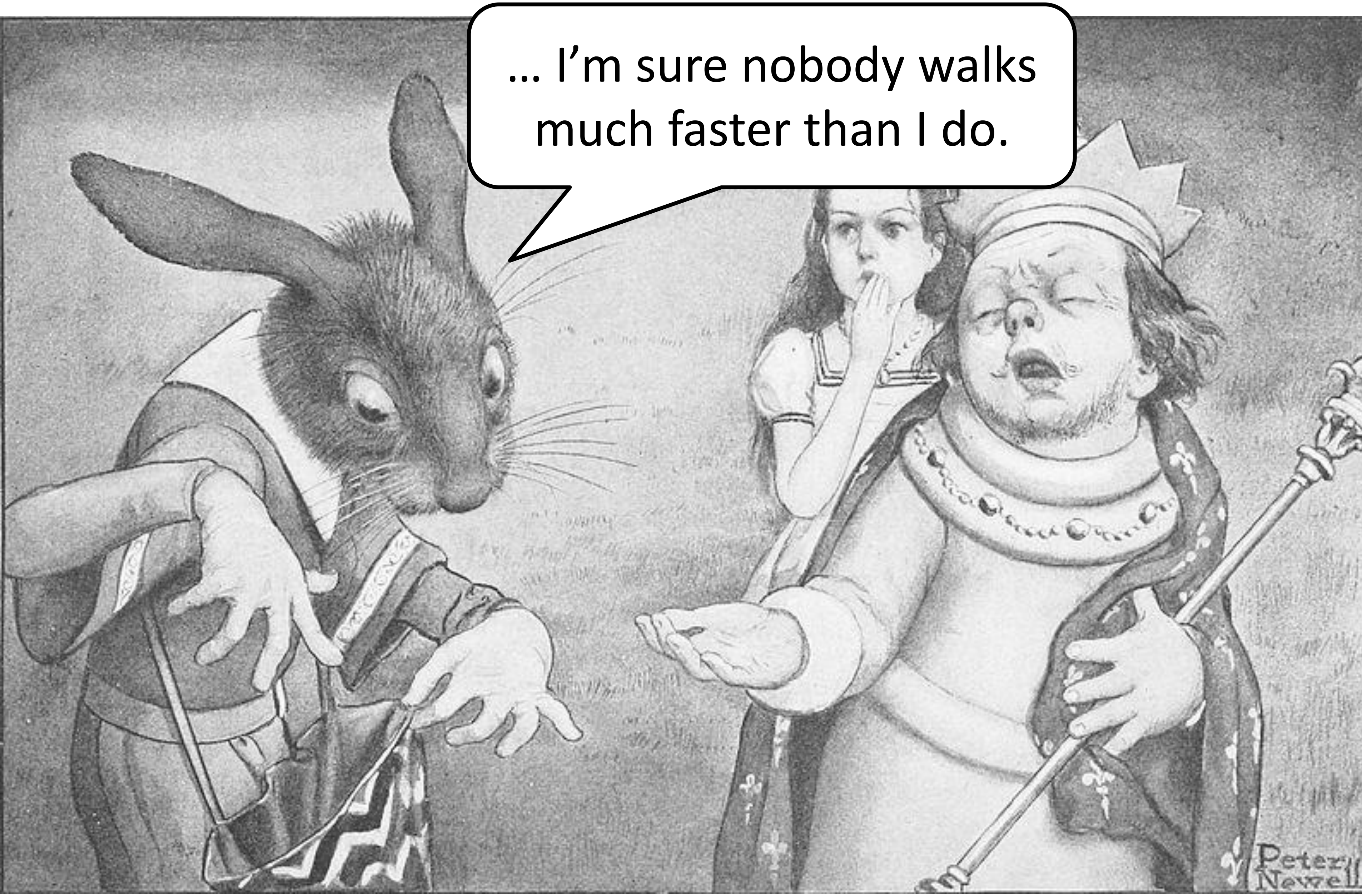


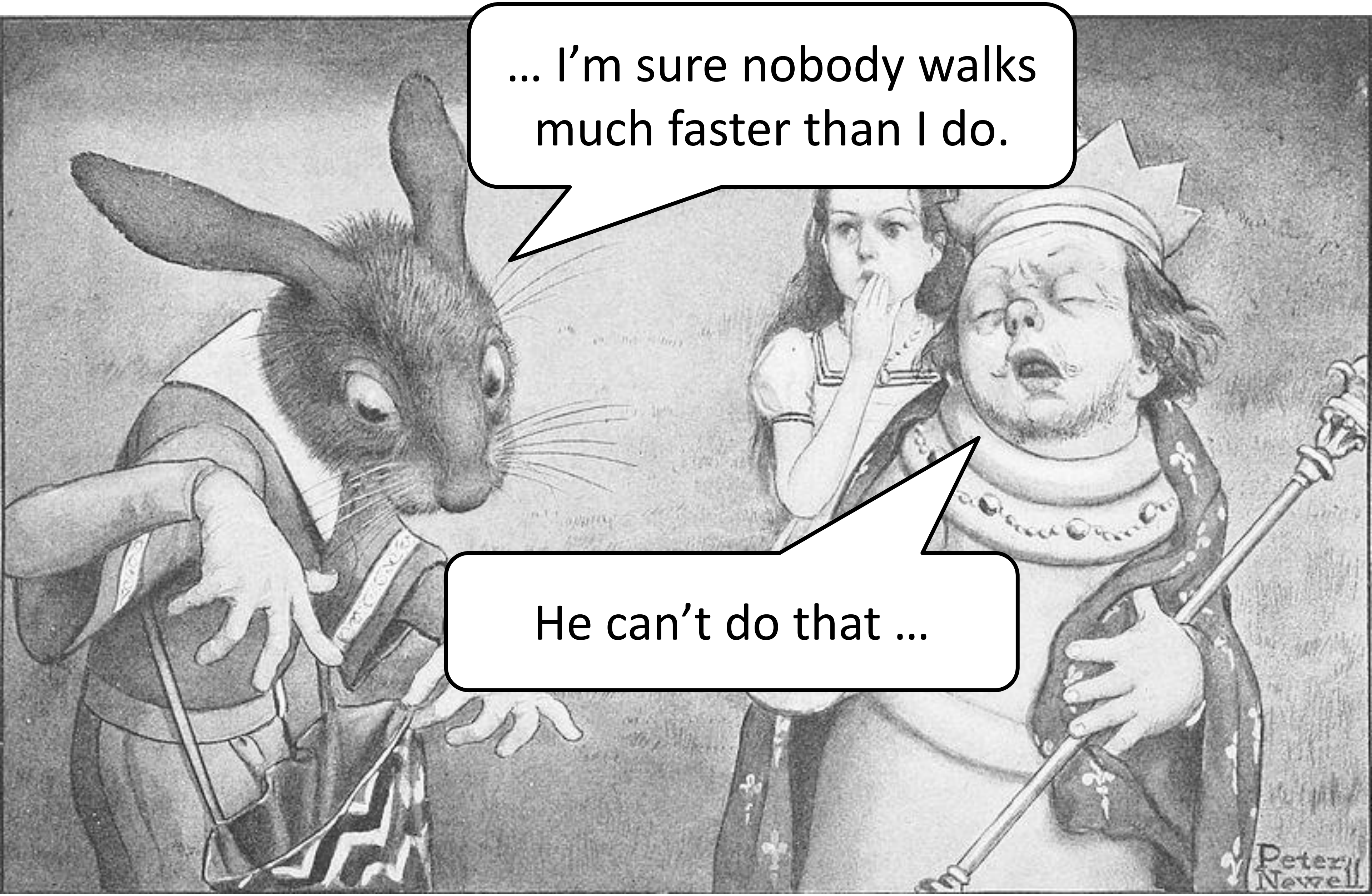

Peter Newell

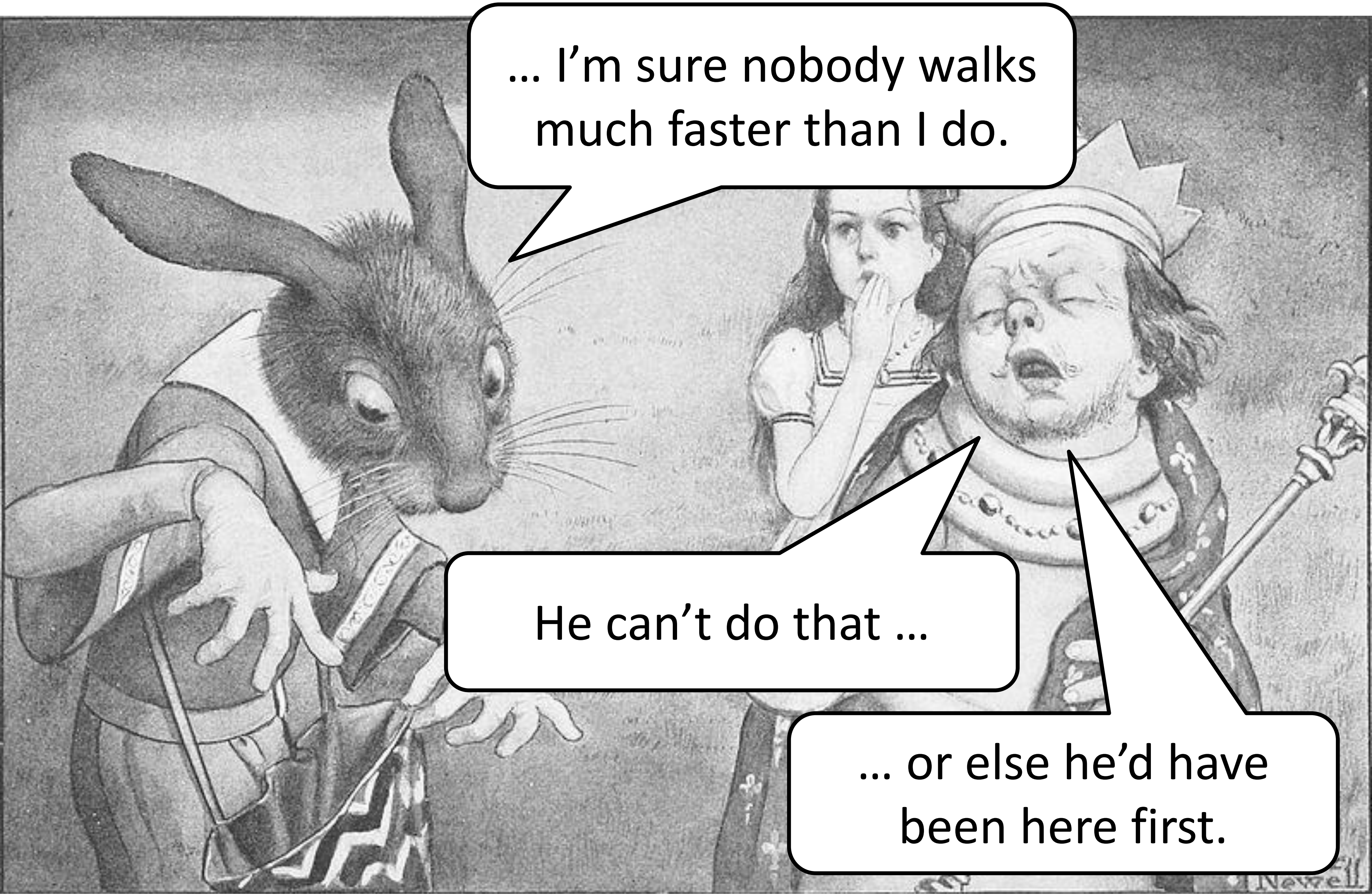

Who did you pass on the road?
Peter Newell


Nobody!
Peter Newell


Nobody!
… So of course Nobody walks slower than you do.
Peter Newell


… I’m sure nobody walks much faster than I do.
Peter Newell


… I'm sure nobody walks much faster than I do.
He can't do that …
Peter Newell


… I’m sure nobody walks much faster than I do.
He can’t do that …
… or else he’d have been here first.

A Brief Word About Nothing

What is going on in the dialogue here? What makes the joke work?

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The king is treating the word "nobody" as a *name*.

A Brief Word About Nothing

What is going on in the dialogue here? What makes the joke work?

The king is treating the word "nobody" as a *name*.

But it *isn't* a name.

A Brief Word About Nothing

What is going on in the dialogue here? What makes the joke work?

The king is treating the word "nobody" as a *name*.

But it *isn't* a name. The word "nothing" does not designate any *thing*.

A Brief Word About Nothing

If *nothing* is not a name, then how should we translate sentences like, “Nobody walks slower than you do”?

A Brief Word About Nothing

If *nothing* is not a name, then how should we translate sentences like, “Nobody walks slower than you do”?

Let W = “… walks slower than …”

A Brief Word About Nothing

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Let W = “… walks slower than …”

Let c name the person indexed by “you.”

A Brief Word About Nothing

If *nothing* is not a name, then how should we translate sentences like, "Nobody walks slower than you do"?

Finally, let P = "… is a person."

A Brief Word About Nothing

If *nothing* is not a name, then how should we translate sentences like, “Nobody walks slower than you do”?

$\sim(\exists x)(Px \land Wxc)$

Next Time

We will talk about validity in first-order logic.