First-Order Logic

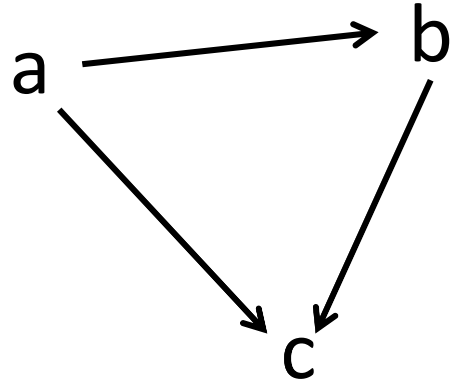
Models & Validity

Review: Relations

Last time, we looked at three properties of relations: reflexivity, symmetry, *and* transitivity. Any relation that has all three properties is called an *equivalence relation*.

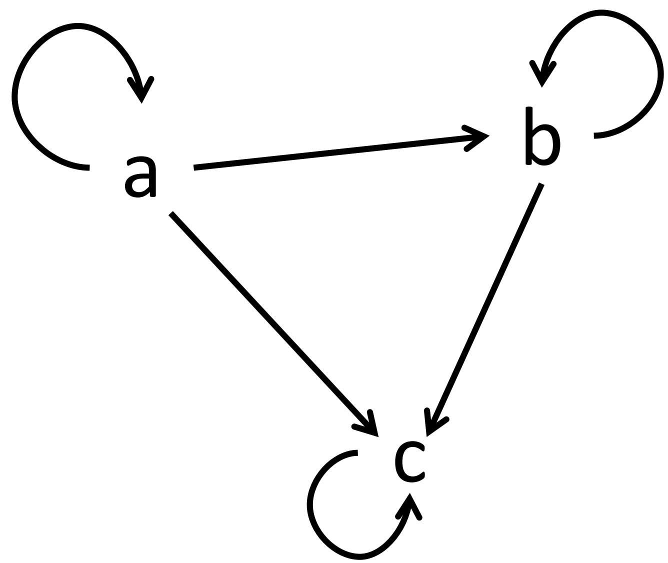
Review: Relations

Suppose we have a relation R over three individuals as pictured below. What properties hold for the relation R?



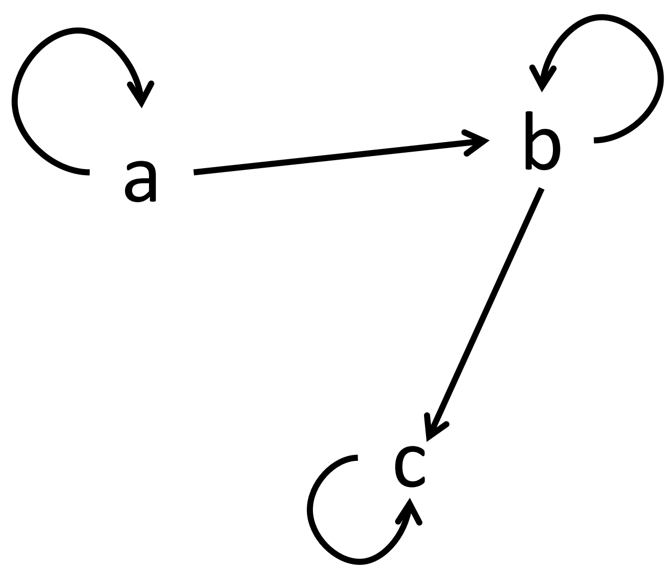
Review: Relations

What properties hold for the relation now?



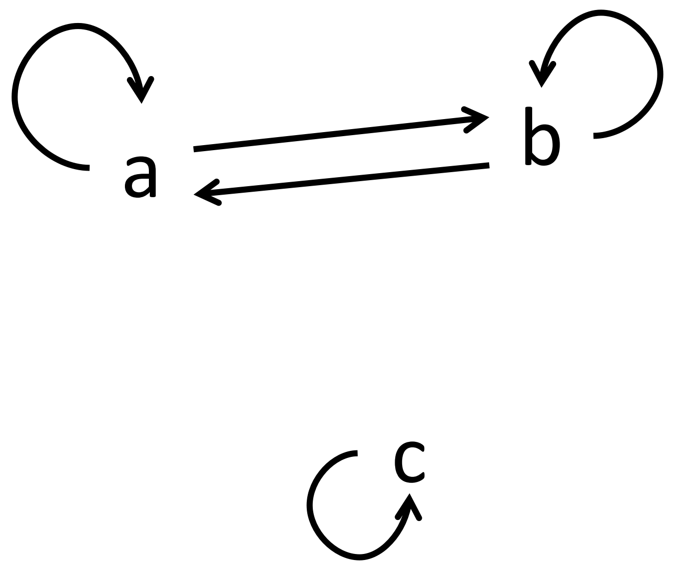
Review: Relations

After removing the a to c edge, what properties hold for the relation?



Review: Relations

Finally, what properties hold for the relation, now?



Review: Relations

Let's do some simple translations. Suppose T = "… is taller than …" and b stands for Betty.

(∃x)Txb

(∀x)Tbx

Everyone is taller than someone or other.

Betty is not taller than herself.

Identity

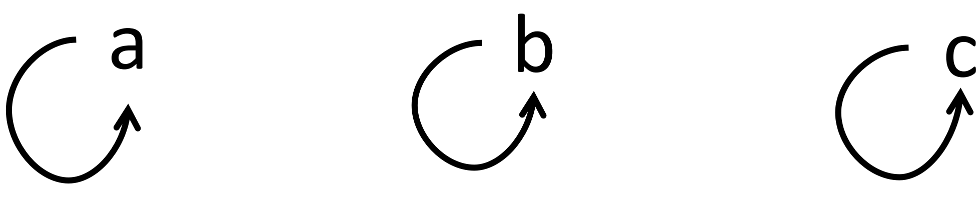
For the most part, we treat relations in a generic way. However, one relation is special.

Identity gets its own symbol, =, and we write (a = b), rather than =ab.

Identity

Identity is an equivalence relation: it is reflexive, symmetric, and transitive.

In fact, identity is the smallest or most fine-grained equivalence relation.



Identity

We can use identity to translate sentences involving superlatives or numerical claims.

Jim is the shortest man in the room.

$$(Mj \wedge Rj) \wedge (\forall x)((Mx \wedge Rx) \rightarrow (Sjx \vee (j = x)))$$

There is exactly one fish.

$$(\exists x)(Fx \wedge (\forall y)(Fy \rightarrow (y = x)))$$

Identity

Let’s try two more examples:

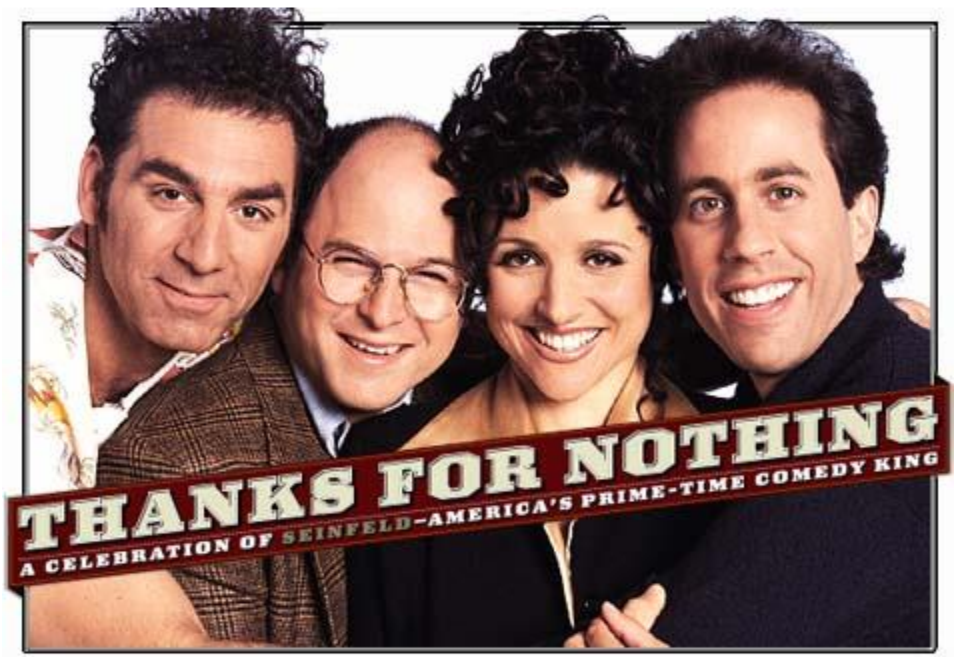
The Godfather was the best film of 1972.

There is exactly one instructor for PHIL 103.

A Brief Word About Nothing

Suppose you want to translate sentences like:

Seinfeld is a show about nothing.



A Brief Word About Nothing

When I want to translate sentences involving words like *nothing*, *nobody*, or *nowhere*, I will generally use the construction $\sim(\exists x)\phi$.

There isn't even one thing that would make ϕ true.

A Brief Word About Nothing

In the *Seinfeld* case, we will let A = "… is about ---" and S = "… is a show." Then let n denote the show *Seinfeld*. Then we can translate the sentence, "*Seinfeld* is a show about nothing," as follows:

$$(Sn \wedge \sim(\exists x)Anx)$$

A Brief Word About Nothing

Lewis Carroll (aka Charles Dodgson) made comic use of nothing in *Through the Looking Glass*.

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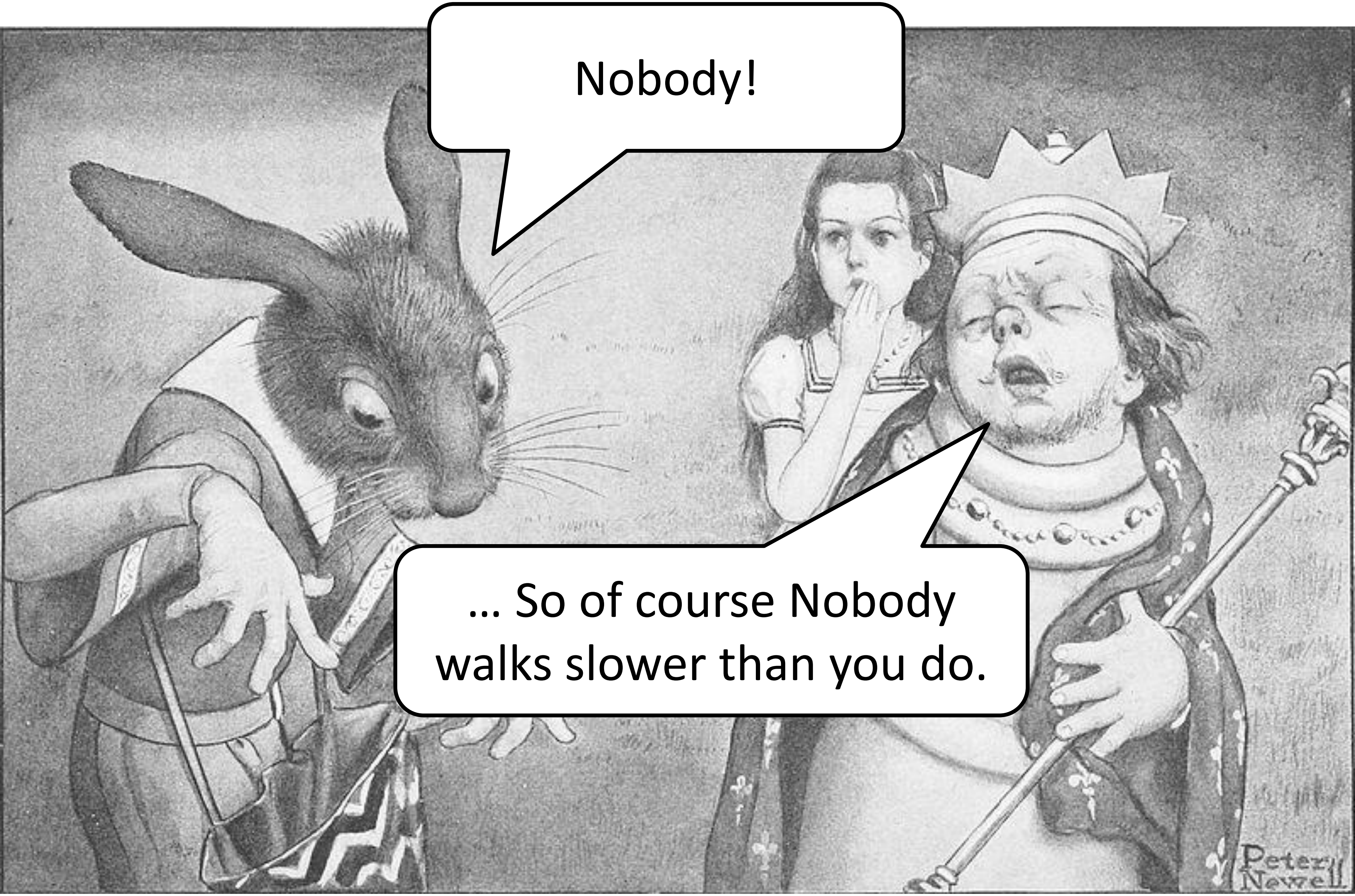
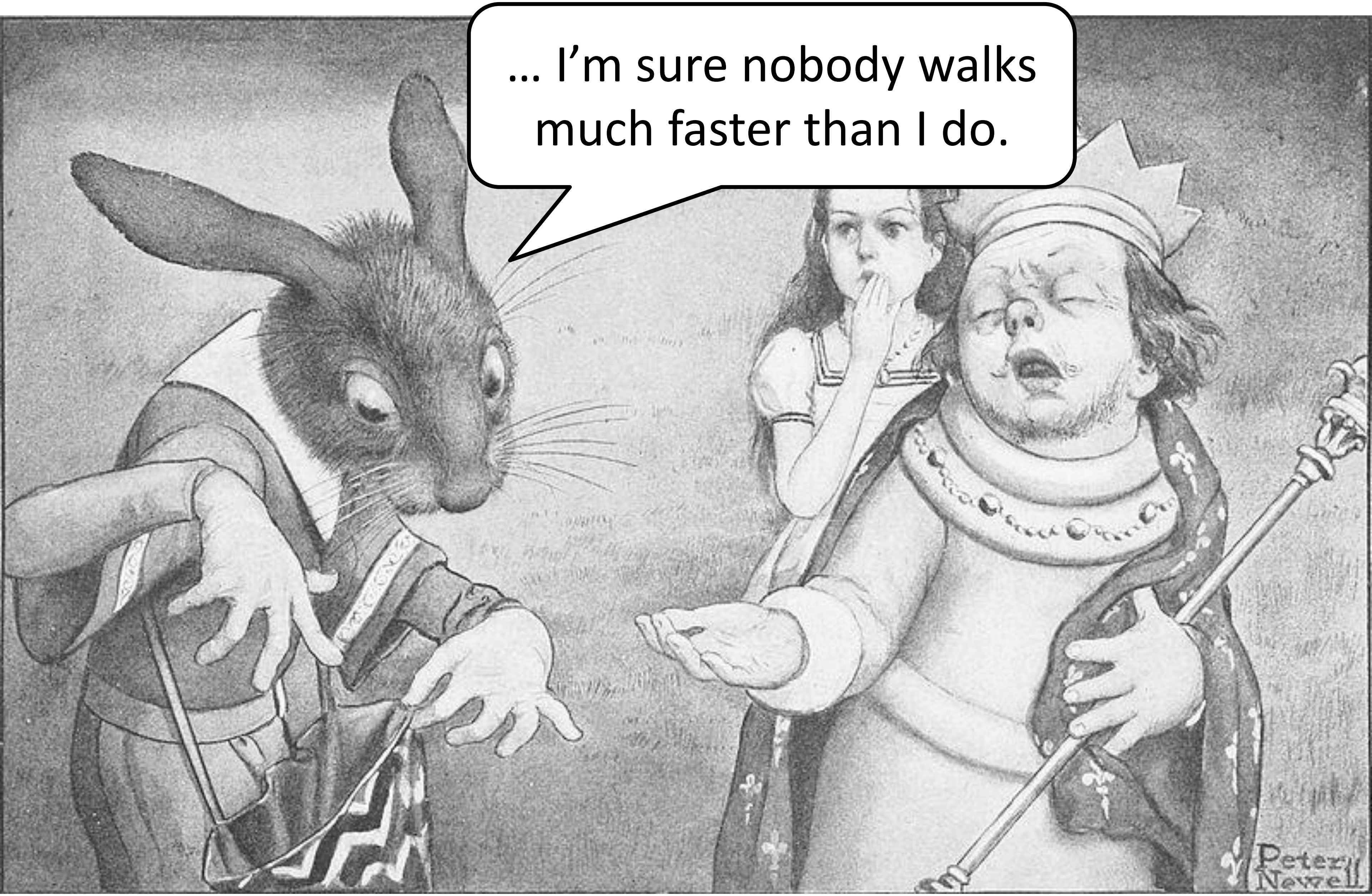
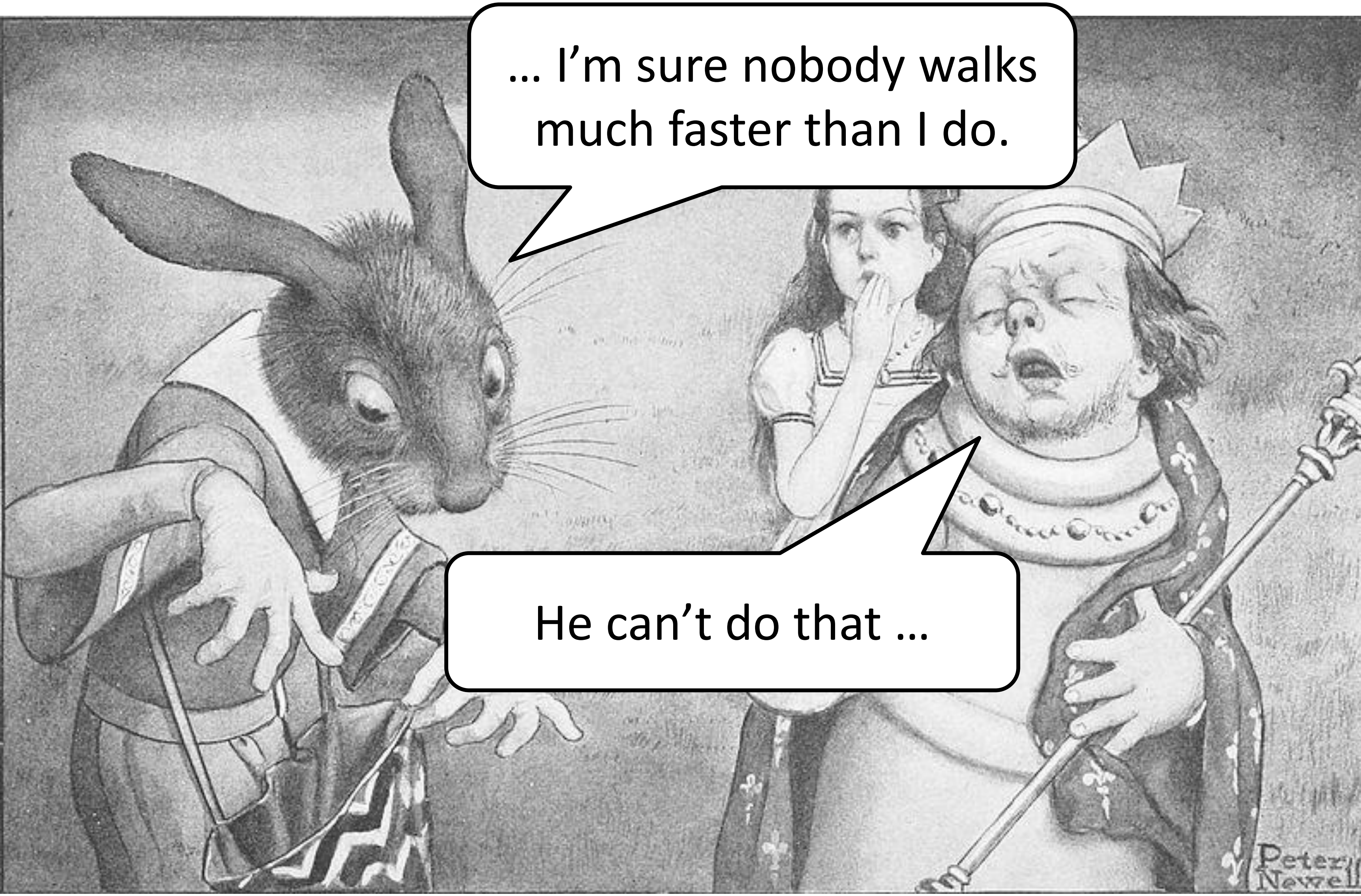
Peter
Newell

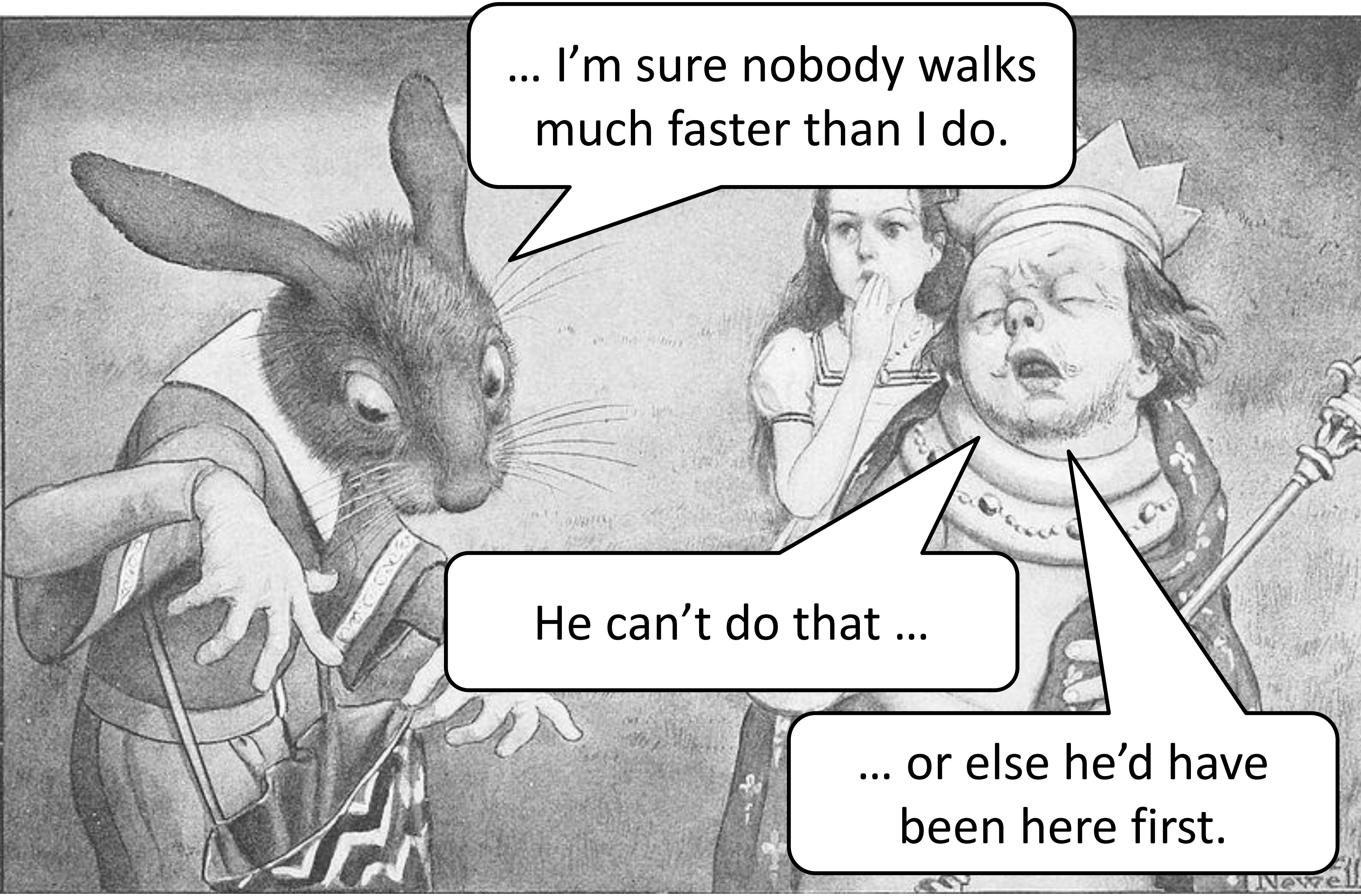

Who did you pass on the road?
Peter Newell


Nobody!
Peter Newell


Nobody!
… So of course Nobody walks slower than you do.
Peter Newell


… I'm sure nobody walks much faster than I do.
Peter Newell


… I’m sure nobody walks much faster than I do.
He can’t do that …
Peter Newell


… I’m sure nobody walks much faster than I do.
He can’t do that …
… or else he’d have been here first.

A Brief Word About Nothing

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The king is treating the word “nobody” as a *name*.

But it *isn’t* a name.

A Brief Word About Nothing

What is going on in the dialogue here? What makes the joke work?

The king is treating the word "nobody" as a *name*.

But it *isn't* a name. The word "nothing" does not designate any *thing*.

A Brief Word About Nothing

If *nothing* is not a name, then how should we translate sentences like, “Nobody walks slower than you do”?

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If *nothing* is not a name, then how should we translate sentences like, “Nobody walks slower than you do”?

Let W = “… walks slower than …”

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Let W = "… walks slower than …"

Let c name the person indexed by "you."

A Brief Word About Nothing

If *nothing* is not a name, then how should we translate sentences like, “Nobody walks slower than you do”?

Finally, let P = “… is a person.”

A Brief Word About Nothing

If *nothing* is not a name, then how should we translate sentences like, “Nobody walks slower than you do”?

$\sim(\exists x)(Px \wedge Wxc)$

Models

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We then used truth tables to test arguments for validity.

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We treated each row in a truth table as a possible world.

Models

Truth tables won't work for us in first-order logic. The possible worlds are too complicated!

In first-order logic, instead of truth tables, we use *small worlds*.

Models

A small world is a *freely chosen* finite collection of constant terms together with their predicates and relations.

Small worlds are free constructions with only one constraint: they have to be non-empty.

Models

A small world is a *freely chosen* finite collection of constant terms together with their predicates and relations.

For example, we might describe a small world in which we have three constants (a, b, and c), two predicates (M and N), and one two-place relation (R).

Models

We can represent predicates and relations by writing down some constant terms and then using closed curves and directed graphs:

a

b

c

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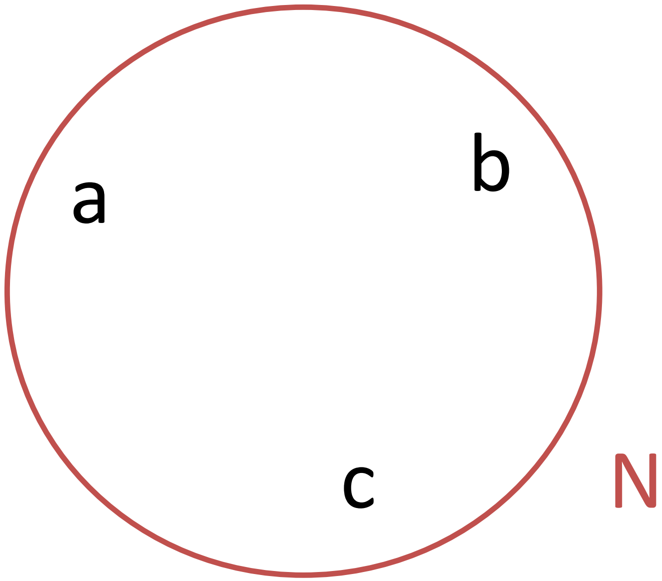

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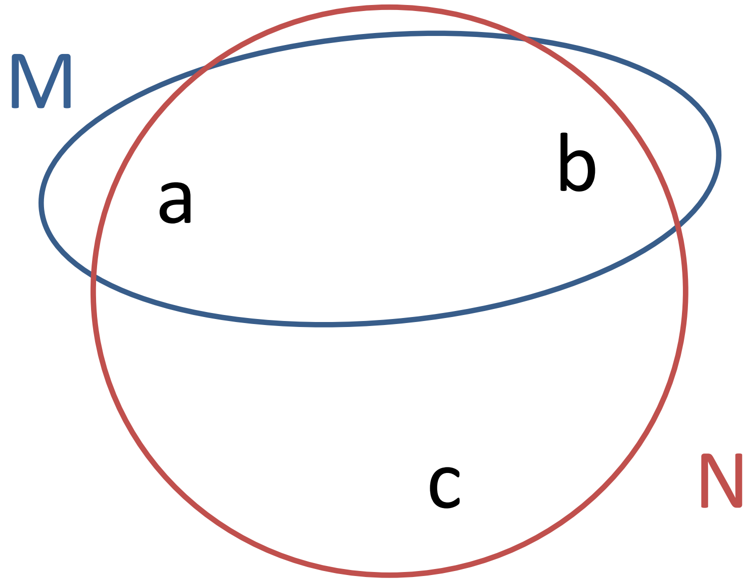

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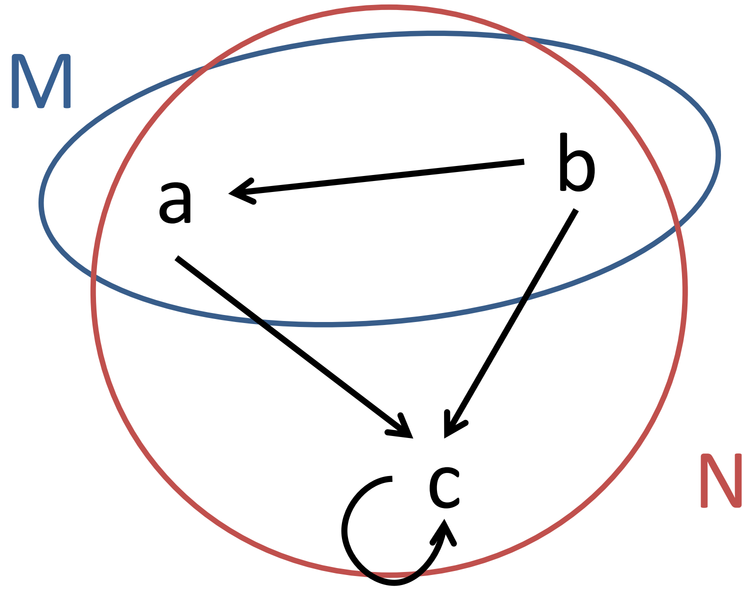


Models

A small world is a free construction in the sense that we specify how the predicates and relations in the small world apply or fail to apply to the constants in that world.

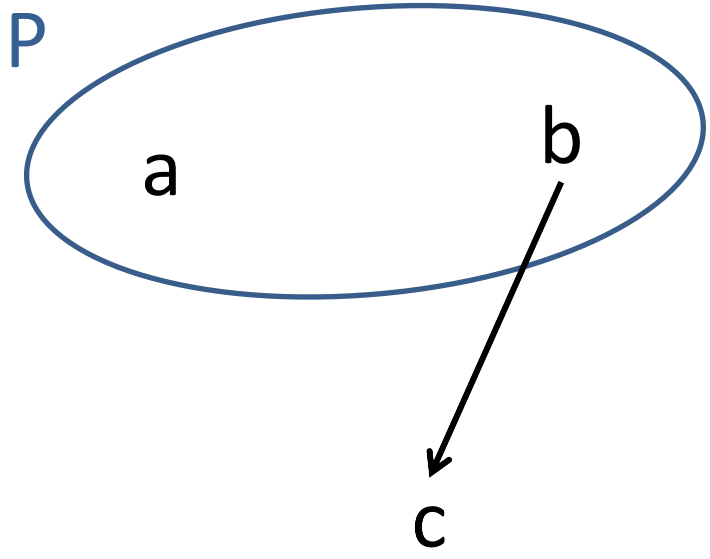
Models

If a sentence is true with respect to a small world, then we say that the small world is a *model* of the sentence.

Let's consider a simple example of a small world.

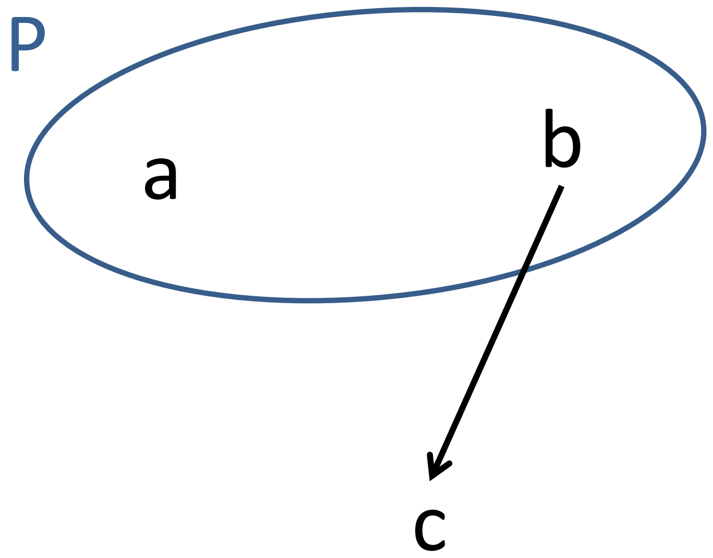
Models

We may represent our example small world with a diagram, like this:



Models

Our example small world is a model of the sentences ∼Pc, (∃x)Px, and (∃x)(∃y)Rxy.



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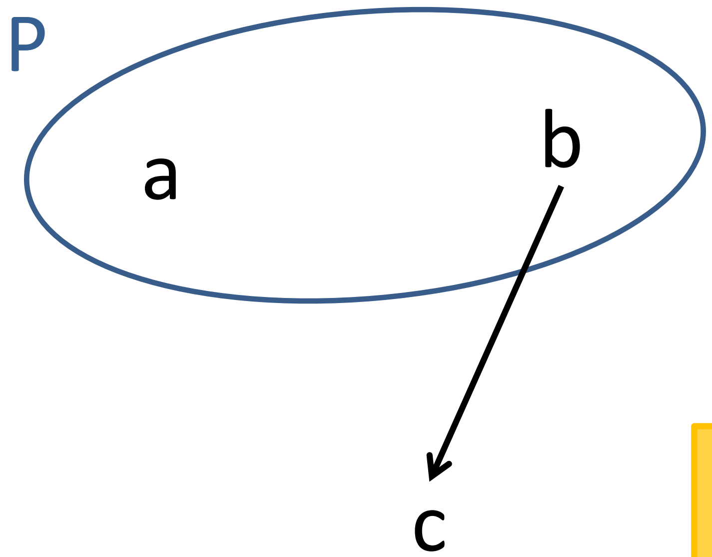

Is it a model of (∃x)(Rbx)?
What about (∀x)(Px)?

Models

We will sometimes use the following notation.
Let $\mathscr{M}$ be a small world, and let ϕ be a sentence.
If $\mathscr{M}$ is a model for ϕ, then we write:

$$\{\ \} \models_{\mathscr{M}} \phi$$

Models

If *every* $\mathscr{M}$ is a model for ϕ, then we remove the subscript, $\mathscr{M}$, and we write:

$$\{\ \} \vDash \phi$$

Models

A small world $\mathscr{M}$ is a model for some *collection* of sentences just in case $\mathscr{M}$ is a model for *every* sentence in the collection.

Validity

We are now going to use small worlds to give an account of validity for first-order logic.

Validity

In zeroth-order logic, we said that an argument is valid if its conclusion is true whenever its premisses are all true.

Our account of validity in first-order logic is very similar.

Validity

Let $\Gamma = \{\phi_1, \ldots, \phi_n\}$ be a collection of sentences. An argument from Γ to ϕ is *valid* in first-order logic just in case for every small world $\mathcal{M}$, if $\mathcal{M}$ is a model of Γ, then $\mathcal{M}$ is a model of ϕ.

$$\Gamma \vDash \phi$$

Validity

If $\Gamma = \{ \}$, also denoted $\varnothing$, then we call ϕ a *logical truth*. Some writers call ϕ a *valid formula*, but we are going to reserve the term "valid" to describe arguments.

$$\varnothing \vDash \phi$$

Validity

Unlike in zeroth-order logic, we do not have a mechanical procedure for checking the validity of an argument in first-order logic. However, we will sometimes construct small worlds to show invalidity.

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Unlike in zeroth-order logic, we do not have a mechanical procedure for checking the validity of an argument in first-order logic. However, we will sometimes construct small worlds to show invalidity.

How would such a construction go?

Validity

Let’s show that the following argument is not valid:

$$\{ (\exists x)Fx \} \vDash (\forall x)Fx$$

Validity

Let's show that the following argument is not valid:

$$\{ (\exists x)Fx \} \models (\forall x)Fx$$

We need to find a small world that is a model of $(\exists x)Fx$ but not a model of $(\forall x)Fx$.

Validity

When looking for an appropriate small world, start with a single constant and work up in number.

a

Validity

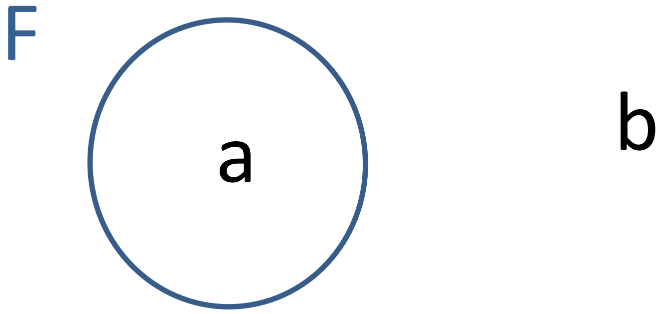
When looking for an appropriate small world, start with a single constant and work up in number.

a

In this case, a single constant isn't enough. So, we'll add a second constant.

Validity

With two constants, we can distinguish between the existential and universal quantifiers.



Next Time

We will start thinking about proof theory for first-order logic.