PHIL 103: Logic and Reasoning QRII

Practice #7 Solutions

1. Show the following: { (∀x)(Fx → Gx) } ⊢ ((∀x)Fx → (∀x)Gx)

1	(1)	(∀x)(Fx → Gx)	A
2	(2)	(∀x)Fx	A (for CP)
1	(3)	(Fa → Ga)	1 ∀E
2	(4)	Fa	2 ∀E
1,2	(5)	Ga	3,5 →E
1,2	(6)	(∀x)Gx	5 ∀I
1	(7)	((∀x)Fx → (∀x)Gx)	2,6 CP

Since no constants appear in either line (1) or line (2), we don't have to worry about the constraints for ∀I.

2. Show the following: { (∀x)(Gx → Hx), (∃x)(Fx ∧ Gx) } ⊢ (∃x)(Fx ∧ Hx)

1	(1)	(∀x)(Gx → Hx)	A
2	(2)	(∃x)(Fx ∧ Gx)	A
3	(3)	(Fa ∧ Ga)	A (for CP)
3	(4)	Fa	3 ∧E
3	(5)	Ga	3 ∧E
1	(6)	(Ga → Ha)	1 ∀E
1,3	(7)	Ha	5,6 →E
1,3	(8)	(Fa ∧ Ha)	4,7 ∧I
1,3	(9)	(∃x)(Fx ∧ Hx)	8 ∃I
1	(10)	((Fa ∧ Ga) → (∃x)(Fx ∧ Hx))	3,9 CP
1,2	(11)	(∃x)(Fx ∧ Hx)	2,10 ∃E

The constant being used is a. The existential elimination step does not violate either of the constraints. To see that, first consider what the form of the rule is here. This application of the rule looks like:

(∃x)(Fx ∧ Gx)
((Fa ∧ Ga) → (∃x)(Fx ∧ Hx))

(∃x)(Fx ∧ Hx)

And in this application, we have ϕ = (Fx ∧ Gx), ψ = (∃x)(Fx ∧ Hx), and ϕ[x/a] = (Fa ∧ Ga). So we have the more abstract rule as:

$(\exists x)\phi$
$(\phi[x/a] \rightarrow \psi)$

ψ

Now, recall our constraints. First, the constant a may not appear in either ϕ or ψ. That's fine: there aren't any constants in ϕ or ψ. Second, the conditional $(\phi[x/a] \rightarrow \psi)$ may not depend on any formula containing the constant a. Again, that's fine: the conditional depends only on the formula in line (1), which contains no constants at all.

3. Show the following: { (∃x)~Fx } ⊢ ~(∀x)Fx

1	(1)	(∃x)~Fx	A
2	(2)	(∀x)Fx	A*
3	(3)	~Fa	A (for CP)
2	(4)	Fa	3 ∀E
	(5)	((∀x)Fx → Fa)	2,4 CP
3	(6)	((∀x)Fx → ~Fa)	3→I
3	(7)	~(∀x)Fx	5,6 ~I
	(8)	(~Fa → ~(∀x)Fx)	3,7 CP
1	(9)	~(∀x)Fx	1,8 ∃E

4. Show the following: { ~(∀x)Fx } ⊢ (∃x)~Fx

1	(1)	~(∀x)Fx	A
2	(2)	~(∃x)~Fx	A*
3	(3)	~Fb	A (for CP)
3	(4)	(∃x)~Fx	3 ∃I
	(5)	(~Fb → (∃x)~Fx)	3,4 CP
2	(6)	(~Fb → ~(∃x)~Fx)	2→I
2	(7)	~~Fb	5,6 ~I
2	(8)	Fb	7 ~E
2	(9)	(∀x)Fx	8 ∀E
	(10)	(~(∃x)~Fx → (∀x)Fx)	2,9 CP
1	(11)	(~(∃x)~Fx → ~(∀x)Fx)	1 →I
1	(12)	~~(∃x)~Fx	10,11 ~I
1	(13)	(∃x)~Fx	12 ~E

5. Prove that identity is an equivalence relation.

In order to prove that identity is an equivalence relation, we need to show that identity is reflexive, symmetric, and transitive. Hence, we need to prove the following:

Reflexive: $(\forall x)(x = x)$
Symmetric: $(\forall x)(\forall y)((x = y) \rightarrow (y = x))$
Transitive: $(\forall x)(\forall y)(\forall z)(((x = y) \wedge (y = z)) \rightarrow (x = z))$

The first one is immediate by =I.

	(1)	$(\forall x)(x = x)$	=I

The second one goes like this:

1	(1)	$(a = b)$	A (for CP)
	(2)	$(\forall x)(x = x)$	=I
	(3)	$(a = a)$	1 ∧E
1	(4)	$(b = a)$	1,3 =E
	(5)	$((a = b) \rightarrow (b = a))$	1,4 CP
	(6)	$(\forall y)((a = y) \rightarrow (y = a))$	5 ∀I
	(7)	$(\forall x)(\forall y)((x = y) \rightarrow (y = x))$	6 ∀I

The third one goes like this:

1	(1)	$((a = b) \wedge (b = c))$	A (for CP)
1	(2)	$(a = b)$	1 ∧E
1	(3)	$(b = c)$	1 ∧E
1	(4)	$(a = c)$	2,3 =E
	(5)	$(((a = b) \wedge (b = c)) \rightarrow (a = c))$	1,4 CP
	(6)	$(\forall z)(((a = b) \wedge (b = z)) \rightarrow (a = z))$	5 ∀I
	(7)	$(\forall y)(\forall z)(((a = y) \wedge (y = z)) \rightarrow (a = z))$	6 ∀I
	(8)	$(\forall x)(\forall y)(\forall z)(((x = y) \wedge (y = z)) \rightarrow (x = z))$	7 ∀I