PHIL 103: Logic and Reasoning QRII

Practice #8

1. Suppose you are a teacher. One day, a student says to you, "I don't have my homework today because my dog ate it." Your prior degree of belief that any given homework will be eaten by a dog is very low: 0.01. You also have credence of 1 that your student would say that a dog ate the homework given that a dog really did eat the homework. And finally, you have credence of 0.5 that your student would say that a dog ate the homework given that a dog *didn't* eat the homework. If you are a personalist Bayesian, what degree of belief should you have that a dog really did eat the homework? Do you think you should *believe* your student? Explain your answers.

Solution. We want to find the probability that a dog ate the homework given the student's testimony. Let D = "A dog ate the student's homework," and let T = "The student says that D." Then we have the following credences:

$\Pr(D) = 0.01$
$\Pr(\sim D) = 1 - \Pr(D) = 0.99$

$\Pr(T \mid D) = 1$
$\Pr(T \mid \sim D) = 0.5$

Putting it all together, we can use Bayes' theorem to calculate $\Pr(D \mid T)$ as follows:

$$\Pr(D \mid T) = \frac{\Pr(T \mid D)\Pr(D)}{\Pr(T \mid D)\Pr(D) + \Pr(T \mid \sim D)\Pr(\sim D)} = \frac{1 \cdot 0.01}{1 \cdot 0.01 + 0.5 \cdot 0.99} \approx 0.0198$$

Hence, by Bayes' rule, our posterior degree of belief should be about twice as strong as it was that a dog ate the student's homework, but it should still only be rated as about a 2% chance.

The connection between degrees of belief and full belief (or belief simpliciter) is a very difficult one. But in this case, the degree of belief is so low that it seems plausible to say that you should *not* believe the student.

2. Suppose you know that your cell phone has a probability 0.01 of dropping a call when it is within five miles of a cell tower but a probability 0.05 of dropping a call when it is at distances beyond five miles from a tower. Further suppose that you think you are equally likely to be within five miles as beyond file miles from any tower. You make a call, but in the middle of your conversation, it is dropped. If you are a personalist Bayesian, what probability should you assign to the claim that you are beyond five miles from a cell tower?

Solution. We want to find the probability that we are more than five miles from the nearest cell tower given that a call was dropped. Let B = "I am further than five miles from the nearest cell tower," and let D = "A call was dropped at this distance." Then we have the following credences:

$\Pr(B) = 0.5$
$\Pr(\sim B) = 1 - \Pr(B) = 0.5$

$\Pr(D \mid B) = 0.05$
$\Pr(D \mid \sim B) = 0.01$

Putting it all together, we can use Bayes' theorem to calculate $\Pr(B \mid D)$ as follows:

$$\Pr(B \mid D) = \frac{\Pr(D \mid B)\Pr(B)}{\Pr(D \mid B)\Pr(B) + \Pr(D \mid \sim B)\Pr(\sim B)} = \frac{0.05 \cdot 0.5}{0.05 \cdot 0.5 + 0.01 \cdot 0.5} \approx 0.83$$

Hence, by Bayes' rule, our posterior degree of belief should be about 0.83 that we are more than five miles from the nearest tower.

3. According to the Census of Fatal Occupational Injuries, the probability that a randomly chosen worker was fatally injured on the job in 2013 was about 32 in 1,000,000. In 2013, men made up approximately 57% of the workforce (by hours) and suffered 93% of fatal injuries. What is the probability that a randomly chosen worker suffered a fatal injury given that the worker was a man?

Solution. We want to find the probability that some worker suffered a fatal injury given that the worker was a man. Let F = "The worker suffered a fatal injury," and let M = "The worker was a man." The problem statement gives us the probability of being a man in the workforce, so we do not need to use the law of total probability in this problem. Instead, we have the following credences:

$\Pr(F) = 32 / 1{,}000{,}000$

$\Pr(M \mid F) = 0.93$

$\Pr(M) = 0.57$

Putting it all together, we can use Bayes' theorem to calculate $\Pr(F \mid M)$ as follows:

$$\Pr(F \mid M) = \frac{\Pr(M \mid F)\Pr(F)}{\Pr(M)} = \frac{32 \cdot 0.93}{0.57 \cdot 1{,}000{,}000} \approx 0.000052$$

In other words, the chance that a worker suffered a fatal injury given that he was a man was about 52 in 1,000,000.

4. The probability that a police officer suffered a fatal injury in 2013 was 106 in 1,000,000. By comparison, the probability that a roofer suffered a fatal injury in 2013 was 387 in 1,000,000. Suppose you learn that a randomly chosen worker suffered a fatal injury. Which hypothesis does the evidence favor: that the worker was a police officer or that the worker was a roofer? Suppose you want to know how likely it is that the randomly chosen person was a police officer. What further fact do you need? Find an estimate of that further fact and calculate the probability.

Solution. Let R = "The person was a roofer," let P = "The person was a police officer," and let F = "The person suffered a fatal injury." According to the law of likelihood, the evidence (F) favors the hypothesis that the randomly chosen person was a roofer, since $\Pr(F \mid R) > \Pr(F \mid P)$.

In order to determine the probability that the person was a police officer given that he or she suffered a fatal injury, we need to know the probability that the person was a police officer and the probability that the person suffered a fatal injury. The second of those is given in Problem #3, but we still need the probability that the person was a police officer. Effectively, we need to know what percentage of employed people in the U.S. are police officers. Then we could use Bayes' Theorem to calculate the probability of being a police officer given a fatal injury as follows:

$$\Pr(P \mid F) = \frac{\Pr(F \mid P)\Pr(P)}{\Pr(F)}$$

The Bureau of Labor Statistics website (http://www.bls.gov/) is a good place to start looking for an estimate of the value of $\Pr(P)$.

5. Suppose I have a fair coin. What is the probability of observing at least three heads in five tosses given that I observe heads on the first toss? What is the probability of observing at least three heads in five tosses given that I observe at least two heads in five tosses?

Solution. For the first question, we want the probability of at least three heads in five tosses given that heads is observed on the first toss. That is equivalent to the probability of seeing at least two heads on four tosses, which can be calculated from the binomial distribution as follows:

$$\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = 0.6875$$

For the second question, let A = "We observe at least three heads in five tosses," and let B = "We observe at least two heads in five tosses." Then Bayes' theorem tells us:

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}$$

We could prove (and it is obvious) that if we observe at least three heads in five tosses, then we observe at least two heads in five tosses. Hence, Pr(B | A) = 1. So, what we really need is to calculate Pr(A) and Pr(B). Those probabilities may be calculated as follows:

$$\Pr(A) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 + \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = 0.5$$

$$\Pr(B) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 + \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = 0.8125$$

Note that Pr(B) could more easily be calculated by subtracting the probability of seeing either zero or one head in five tosses from one. Check that you get the same answer either way.

After dividing we find: Pr(A | B) = 0.5 / 0.8125 ≈ 0.615.

6. Suppose the chances of being struck by lightning on any given day are 1 in 1,000,000 and four times that if it is raining. Suppose that this week, there is a 60% chance that it will rain on Saturday and a 30% chance that it will rain on Sunday. Suppose further that the probability of being struck on any given day is the same as the conditional probability of being struck given that one was not struck on the previous day. What is the probability of either being struck on Saturday or not being struck on Saturday and being struck on Sunday?

Solution. Let S_1 = "Struck by lightning on Saturday," let S_2 = "Struck by lightning on Sunday," and let R = "It rains." Then the probability of either being struck on Saturday or not being struck on Saturday and being struck on Sunday is equal to the sum of the probability of being struck on Saturday and the product of the probability of not being struck on Saturday and the probability of being struck on Sunday. By the law of total probability, the probability that you will be struck by lightning on Saturday is given by:

$$\Pr(S_1) = \Pr(S_1 \mid R)\Pr(R) + \Pr(S_1 \mid {\sim}R)\Pr({\sim}R) = \frac{1}{1{,}000{,}000}\cdot\frac{4}{10} + \frac{4}{1{,}000{,}000}\cdot\frac{6}{10}$$

$$= 0.0000028$$

And hence, the probability of not being struck on Saturday is 1 – Pr(S_1) = 0.9999972

Similarly, the probability of being struck by lightning on Sunday is given by:

$$\Pr(S_1) = \Pr(S_2 \mid R)\Pr(R) + \Pr(S_2 \mid {\sim}R)\Pr({\sim}R) = \frac{1}{1{,}000{,}000}\cdot\frac{7}{10} + \frac{4}{1{,}000{,}000}\cdot\frac{3}{10}$$

$$= 0.0000019$$

Consequently, the probability we want to calculate is 0.0000028 + 0.9999972 × 0.0000019, which is approximately 0.0000047.

7. Suppose you are getting on the bus in the morning. To your surprise all ten of the seats on the left side of the bus are filled, while all ten of the seats on the right side are empty. Assuming that passengers sit down one at a time and that each passenger selects his or her seat at random from the available ones, what is the probability that this configuration would occur? Give two reasons why you wouldn't expect to observe such an arrangement in real life.

Solution. One straightforward way to calculate the desired probability is to observe that the first passenger has a choice of 20 seats and 10 of those seats satisfy the condition. Since each of the seats is equally likely to be chosen, the probability of satisfying the condition is 10/20. Given the choice that the first passenger makes, the second passenger has only 19 available options and only 9 of those satisfy the condition. But given that restricted space, the choice is not further affected by the first passenger's choice. Hence, we multiply 10/20 by 9/19. We continue in this way until the tenth passenger, who has 11 options but only one that satisfies the condition. Hence, the probability is given by:

$$\frac{10}{20}\cdot\frac{9}{19}\cdot\ldots\cdot\frac{1}{11}\approx 0.0000054$$

Not only is the arrangement very unlikely given the modeling assumption, the modeling assumption is unlikely to be true of the actual world – and the way it is false about the world actually favors seeing the strange result! After all, people normally do not sit next to strangers if there are free seats that are not next to anyone.

8. Given the same modeling assumptions as in Problem #7, what is the probability that the first ten passengers all sit on the left side of the bus given that the first five passengers all sit on the left side of the bus?

Solution. Let A = "The first ten all sit on the left," and let B = "The first five all sit on the left." As in Problem #5, applying Bayes' theorem shows that the probability we want to calculate reduces to the following:

$$\Pr(A\,|\,B)=\frac{\Pr(A)}{\Pr(B)}$$

We calculated the probability that the first ten all sit on the left in Problem #7. The probability that the first *five* all sit on the left may be calculated in a similar way as follows:

$$\frac{10}{20}\cdot\frac{9}{19}\cdot\frac{8}{18}\cdot\frac{7}{17}\cdot\frac{6}{16}\approx 0.016$$

Hence, the desired probability is approximately 0.00033.